RADIATION-CONDUCTION HEAT TRANSFER IN A THIN SEMITRANSPARENT CYLINDER IN THE LIGHT-GUIDE APPROXIMATION

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The calculation of the temperature and radiation intensity in a region at the boundary of which multiple reflection and refraction of light is possible is, in general, a complex and little-investigated problem in the theory of complex heat transfer. It is therefore natural that we should wish to consider models which, without losing sight of the main features of the phenomena and remaining of practical value, enable one to reduce the mathematical difficulties to reasonable limits.

The traditional problem of radiant heat transfer is a problem of a plane layer in which the temperature and radiation intensity vary only over its thickness [1, 2]. Another type of problem arises when investigating heat transfer in thin regions when the mean temperature variation occurs along the region, while transverse to the region the temperature can be regarded as practically constant. Such a "one-dimensional" approach has been known for a long time in the theory of heat conduction. However, in the theory of radiation-conduction heat transfer it has only recently been developed, although the problem in this formulation is not only of theoretical but also of considerable practical interest, for example, when extracting semitransparent crystals from a melt using the Czochralski or Stepanov method.

In this paper we consider heat transfer in a thin circular cylinder surrounded by a cylindrical screen with an axisymmetrical temperature distribution. The gap between the screen and the cylinder is evacuated or filled with a gas having a low absorption. It is assumed that the following relations are satisfied:

$$d/z \ll 1,\ kd \ll 1,\ a/z \ll 1, \tag{1}$$

where d is the diameter of the cylinder, z is its length, k is the absorption coefficient, and a is the distance between the cylinder and the screen. Suppose the side surface of the cylinder is transparent and reflecting, and the conductivity is isotropic. We will also assume that there is no scattering and the absorption coefficient is independent of the temperature and frequency. One base of the cylinder (which we will call its bottom) is assumed to be heated and absolutely black, while the other is cold and diffusely reflecting. Note that when drawing a crystal from a melt the bottom corresponds to the crystallization front.

The heat-conduction equation and the boundary conditions for it can be written in the form

$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(\lambda r\frac{\partial T}{\partial r}\right)=4n^2k\sigma T^4-k\int\limits_{4\pi} i\,(x, r, \Omega)\,d\Omega, \tag{2}$$

$$\begin{aligned}&\text{for}\quad x=0\qquad T=T_0,\\ &\text{for}\quad x=z\qquad T=T_1;\\ &\text{for}\quad r=R\qquad \partial T/\partial r=0,\end{aligned} \tag{3}$$

where n is the refractive index, σ is Stefan's constant, and i is the radiation intensity. Boundary condition (3) looks somewhat unusual. Unlike the corresponding condition for a radiating nontransparent cylinder it does not contain radiation terms. This is quite natural, however, since semitransparent materials radiate over the whole volume and, consequently, the radiation transfer of energy is taken into account in the equation itself and not in the boundary conditions.

The intensity of the radiation i is given by the equation

$$di/ds+ki=n^2\sigma T^4/\pi,$$

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where s is the length of the path along the light beam. Hence, in accordance with the above assumptions,

$$i = \frac{n^2\sigma T_0^4}{\pi} \quad \text{for} \quad x = 0,$$

$$i^- = \frac{\alpha}{2\pi}\int i^+ d\Omega \quad \text{for} \quad x = z,$$

where α is the reflection coefficient.

The intensity of the radiation is made up of the radiation of the bottom of the cylinder i_f, of the cylinder itself i_c, of the screen i_e, and the radiation i_r reflected from the second cold base. Correspondingly

$$I = \int_{4\pi} i d\Omega = I_f + I_c + I_e + I_r.$$

We will first consider the terms I_f and I_c. Figure 1 shows as an example beams arriving at an arbitrary point of the cylinder from arbitrary point-sources on its bottom. For clarity the receiving point is on the axis of the cylinder so that all the rays lie in its meridional plane. It is obvious that rays traveling from the bottom and incident on the side surface of the cylinder at an angle less than the angle of total internal reflection φ_i will leave the cylinder after a certain number of reflections so that at distances of the order of $O(d \tan\varphi_i)$ the radiation from the bottom will contain only those rays which have suffered total internal reflection from the side surface. A similar phenomenon occurs for the radiation I_c of the cylinder itself. That part which arrives at the reception point at angles less than φ_i has a spatial radiation density of the order of O(d). Hence, for a thin cylinder in the radiation from its bottom and from the cylinder itself we need only take into account that part of the energy which propagates along the cylinder as if along a light conductor.

In order to obtain expression for the light-guide part of the radiation it is best to change from a consideration of multiply reflected beams arriving at the reception point from a point source to the direct rays, but which come from a point image of this source. Then in the expressions for the spatial radiation density we can replace the integration over the volume of the cylinder and over its base by integration over an extended region covering the images of all those point sources the rays from which, when arriving at the receiving point, intersect the side surface at an angle greater than φ_i (see Fig. 1). In this case it is convenient to use local cylindrical coordinates connected with the reception point.

Neglecting, due to the smallness of their diameter, the effect of the change in temperature over the cross section of the cylinder on the radiation intensity, we obtain

$$I_f = 4\sigma n^2 T_0^4 F_1(x, r), \quad I_c = 4\sigma n^2 k \int_0^z F_2(|x - x'|, r)\, T^4(x')\, dx',$$

where

$$F_1(x, r) = \frac{1}{4\pi}\int_0^{2\pi} d\varepsilon \int_0^{\rho_1(x)} \frac{\exp\left(-k\sqrt{x^2+\rho^2}\right)}{(x^2+\rho^2)^{3/2}}\, x\rho d\rho; \tag{4}$$

$$F_2(|x - x'|, r) = \frac{1}{4\pi}\int_0^{2\pi} d\varepsilon \int_0^{\rho_1(|x-x'|)} \frac{\exp\left(-k\sqrt{(x-x')^2+\rho^2}\right)}{(x-x')^2+\rho^2}\, \rho d\rho, \tag{5}$$

$$\rho_1(z) = \sqrt{n^2-1}\,\frac{z}{\left(1 - n^2\frac{r^2}{R^2}\sin^2\varepsilon\right)^{1/2}}.$$

It can be seen that the boundary of the region $\rho_1(z)$ depends very much on the radial coordinate r of the receiving point, and when $r > R/n$ there is a range of values of the angles ε for which $\rho_1(z)$ becomes infinite, which corresponds to the presence in the light-guide radiation of rays of infinitely long path length from the point source to the receiving point. This leads to considerable nonuniformity in the spatial radiation density over the cross section of the cylinder – the so-called "edge effect" known from fiber optics [3].

Carrying out the integration over the angle ε in Eq. (4) we obtain

$$F_1(x, r) = F_1(x, 0) + \frac{1}{\pi}\int_n^{bn} \frac{\exp(-kxt)}{t^2}\left(\frac{\pi}{2} - \arcsin\left(\frac{R}{nr}\sqrt{\frac{t^2-n^2}{t^2-1}}\right)\right) dt,$$

where $F_1(x, 0) = \frac{1}{2}\exp(-kx) - \frac{1}{2}\exp(-knx) - \frac{kx}{2}(Ei(kx) - Ei(knx))$

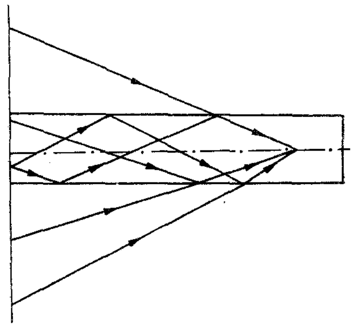

Fig. 1

is the radiation density from the bottom on the axis of the cylinder;

$$b=\frac{\sqrt{1-r^2/R^2}}{\sqrt{1-n^2r^2/R^2}};\quad Ei(x)=\int_x^\infty \frac{e^{-t}}{t}\,dt.$$

In Fig. 2 we show the distribution $F_1(x, r)$ over the cross section of the cylinder for several values of kx for the case when $n = 1.75$ (sapphire). It can be seen that in the region of the side surface the radiation density may be 1.5 times greater than the corresponding value on the cylinder axis.

We will obtain the mean values of F_1 and F_2 over the cross section. Integrating (4) and (5) over the radius and changing the order of integration twice, we obtain

$$F_{1c}(x)=\frac{1}{2}\int_1^\infty \frac{\exp(-tkx)}{t^2}(1-\varphi(t))\,dt,\qquad F_{2c}(z)=-\frac{1}{k}\frac{dF_{1c}(z)}{dz},$$

$$\varphi(t)=\begin{cases}0 & \text{for } t\leqslant n,\\ \frac{2}{\pi}\left(\arcsin\left(\frac{1}{n}\sqrt{\frac{t^2-n^2}{t^2-1}}\right)+\frac{\sqrt{n^2-1}}{n^2}\frac{t\sqrt{t^2-n^2}}{t^2-1}\right) & \text{for } t>n.\end{cases}$$

It is interesting to note that when $n = 1.75$, $F_{1c} = 0.336$ and, consequently, the light-guide part of the radiation from the bottom of the cylinder represents 66% of the total radiation concentrated in the hemisphere. It can also be seen that

$$\lim_{n\to 1} F_{1c}=0,\ \text{and}\ \lim_{n\to\infty} F_{1c}=\frac{1}{2}(e^{-kx}-kxEi(kx)),$$

i.e., it is equal to the radiation from an infinite plane.

In order to obtain the one-dimensional approximation for Eq. (2), we will integrate it over the cross section of the cylinder. Introducing the dimensionless variables (to simplify the notation the bar will be omitted henceforth)

$$\bar{x}=\frac{x}{z},\quad \Theta=\frac{T}{T_0},\quad \bar{I}=\frac{I}{4\sigma n^2T_0^4},\quad \bar{\lambda}=\frac{\lambda}{\lambda_0}$$

and putting

$$m=\frac{4\sigma n^2kT_0^3}{\lambda_0},$$

we obtain an equation which is similar in structure to the equation for a plane layer

$$\frac{d}{dx}\left(\lambda(\Theta)\frac{d\Theta}{dx}\right)=m\left(\Theta^4-F_{1c}(x)-k\int_0^1 F_{2c}(|x-x'|)\,\Theta^4(x')\,dx'-I_r-I_e\right),\tag{6}$$

for $x = 0$, $\Theta = 1$, and for $x = 1$, $\Theta = \Theta_1$.

To integrate this equation it is necessary to specify the radiation from the screen and to determine the reflected radiation from the cold cone of the cylinder. The latter is easily obtained by taking into account once again only the light guide part of the reflected rays. Then

$$I_r(x)=\alpha F_{1c}(1-x)(I_f+I_c+I_e)|_{x=1}.$$

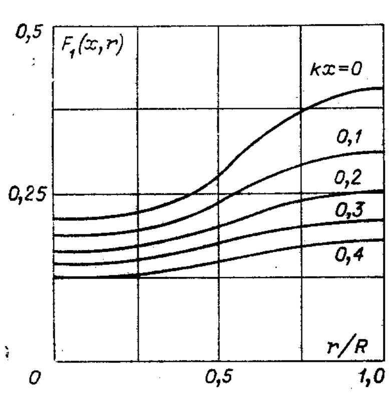

Fig. 2

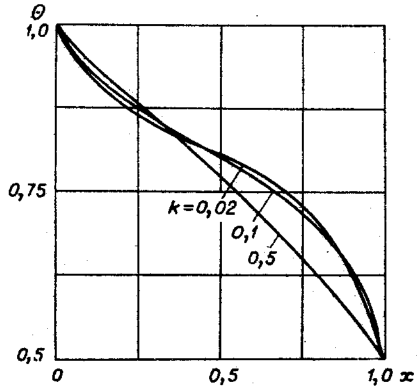

Fig. 3

The situation is more complex when I_e is specified. Determining it rigorously involves considerable computational difficulties due to the combination of refraction and multiple reflection on the side surface of the cylinder. However, in the problem considered we can obtain an approximate expression for the screen radiation by the following discussion.

Suppose the cylinder and the screen are infinitely long and the whole system is at a constant temperature T. Then, as follows from (6),

$$\Theta^4 - 2F_{1c}(0)\Theta^4 - I_e = 0,$$

i.e., the energy losses by radiation in an arbitrary section of the cylinder is compensated by the light-guide radiation $2F_{1c}(0))\Theta^4$ and the radiation of the screen. Hence

$$I_e = (1 - 2F_{1c}(0))\Theta^4. \tag{7}$$

Bearing assumptions (1) in mind we can, firstly, neglect the effect of the cylinder on the screen radiation, and, secondly, assume that the screen heats locally. Expression (7) will then hold in general if instead of T we substitute the local temperature of the screen T_e.

For large absorption coefficients we can obtain the Rosseland approximation for the light-guide flux. Then, everywhere, in addition to the boundary regions, Eq. (6) will have the form

$$\frac{d}{dx}\left((\lambda + \lambda_1)\frac{d\Theta}{dx}\right) = m\,(1 - 2F_{1c}(0))\,(\Theta^4 - \Theta_e^4)\,,$$

$$\lambda_1 = 32\,\frac{\sigma n^2 T_0^3}{k}\left(\frac{1}{6} - \frac{1}{\pi n}\int_n^\infty \frac{\varphi(t)}{t^4}\,dt\right).$$

As an example we will consider a sapphire rod 10 cm long. The thermal conductivity can be approximated by the equation

$$\lambda = \begin{cases} \lambda_0(8.42 - 21.9\Theta + 16.2\Theta^2) & \text{for } \Theta < 0.676, \\ \lambda_0 & \text{for } \Theta \geqslant 0.676, \end{cases}$$
$$\lambda_0 = 0.01 \text{ cal/sec}\cdot\text{cm}\cdot\text{degree.}$$

The temperature variation of the screen can be assumed to be linear:

$$\Theta_e = 1 - 0.5x,$$

while T_0 is equal to the melting point of sapphire (2326°K). In view of the fact that there is no data available for the absorption coefficient of sapphire at such high temperatures the calculations were carried out for k = 0.02, 0.1, and 0.5. The results obtained are shown in Fig. 3.

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